A Few More Index Calculus Algorithms For The Elliptic Curve Discrete Logarithm Problem

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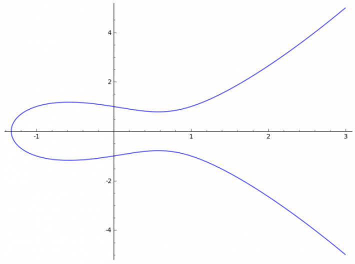
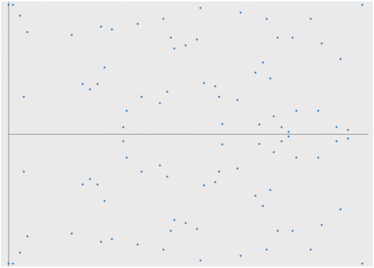
What is the Elliptic Curve Discrete Logarithm Problem?

Elliptic curve in short Weierstrass form: $E : y^2 = x^3 + Ax + B$ over finite field. We will focus on prime order fields.





ECDLP: Given points $P, Q \in E(\mathbb{F}_q)$, find s such that $Q = sP$ (if it exists).

Index Calculus Algorithm for ECDLP (Gaudry)

Let $P, Q \in E(\mathbb{F}_q)$ such that there exists s that solves $Q = sP$.

Factor Base

Define a factor base $\mathcal{F} \subseteq E(\mathbb{F}_q)$.

Point Decomposition $\Rightarrow$ Relation

Generate points $R = aP + bQ$, where a, b are random, and try to write $R = P_1 + P_2 + \ldots + P_m$, where $P_1, \ldots, P_m \in \mathcal{F}$.

Linear Algebra Step

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} log(P_1) \\ log(P_2) \\ \cdots \\ log(P_{|\mathcal{F}|}) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ \cdots \\ a_k \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_k \end{pmatrix} log(Q)$$

Find $v = (v_1, \ldots, v_k)$ such that $vM = 0$, $\sum_i a_i v_i \neq 0$ and $\sum_i b_i v_i$ is invertible. Then $s = (\sum_i a_i v_i)/(\sum_i b_i v_i)$.

Summation Polynomials

Let E be an elliptic curve over a field K.

Definition

For $m \geq 2$, define the m$^{\text{th}}$ summation polynomial $S_m = S_m(X_1, X_2, \ldots, X_m)$ of E by the following property:

Let $x_1, x_2, \ldots, x_m \in \overline{K}$, then $S_m(x_1, x_2, \ldots, x_m) = 0$

if and only if

$\exists y_1, y_2, \ldots, y_m \in \overline{K}$ such that $(x_i, y_i) \in E(\overline{K})$ $\forall i$ and $(x_1, y_1) + (x_2, y_2) + \ldots + (x_m, y_m) = \mathcal{O}$.

Summation Polynomials

Theorem (I. Semaev)

Let E be an elliptic curve given by $y^2 = x^3 + Ax + B$ over a field K with characteristic $\neq 2, 3$.
Then the summation polynomials are given by

$S_2(X_1, X_2) = X_1 - X_2$,

$S_3(X_1, X_2, X_3) = (X_1 - X_2)^2 X_3^2 - 2((X_1 + X_2)(X_1 X_2 + A) + 2B)X_3 + ((X_1 X_2 - A)^2 - 4B(X_1 + X_2))$,

$S_m(X_1, \ldots, X_m) =$
$Res_Y(S_{m-k}(X_1 \ldots, X_{m-k-1}, Y), S_{k+2}(X_{m-k} \ldots, X_m, Y))$
for $m \geq 4$ and any $m - 3 \geq k \geq 1$.

Furthermore, the polynomials S_m, $m \geq 3$, are symmetric and of degree 2^{m-2} in each variable, and absolutely irreducible.

4th Summation Polynomial S_4

Elliptic Curve defined by $y^2 = x^3 + 3x + 3$ over $\mathbb{F}_5$

```
X1^4*X2^4*X3^4 + X1^4*X2^4*X3^3*X4 + X1^4*X2^4*X3^2*X4^2 + X1^4*X2^4*X3*X4^3 + X1^4*X2^4*X4^4 + X1^4*X2^3*X3^4*X4 + 4*X1^4*X2^3*X3^3*X4^2 +
3*X1^4*X2^3*X3^3 + 4*X1^4*X2^3*X3^2*X4^3 + 2*X1^4*X2^3*X3^2*X4 + X1^4*X2^3*X3^2 + X1^4*X2^3*X3*X4^4 + 2*X1^4*X2^3*X3*X4^2 + 3*X1^4*X2^3*X3*X4 +
3*X1^4*X2^3*X4^3 + X1^4*X2^3*X4^2 + X1^4*X2^2*X3^4*X4^2 + 4*X1^4*X2^2*X3^3*X4^3 + 2*X1^4*X2^2*X3^3*X4 + X1^4*X2^2*X3^3 + X1^4*X2^2*X3^2*X4^4 +
2*X1^4*X2^2*X3^2*X4^2 + 2*X1^4*X2^2*X3^2*X4 + 4*X1^4*X2^2*X3^2 + 2*X1^4*X2^2*X3*X4^3 + 2*X1^4*X2^2*X3*X4^2 + X1^4*X2^2*X3*X4 + 4*X1^4*X2^2*X3 +
X1^4*X2^2*X4^3 + 4*X1^4*X2^2*X4^2 + 4*X1^4*X2^2*X4 + 4*X1^4*X2^2 + X1^4*X2*X3^4*X4^3 + X1^4*X2*X3^3*X4^4 + 2*X1^4*X2*X3^3*X4^2 + 3*X1^4*X2*X3^3*X4 +
2*X1^4*X2*X3^2*X4^3 + 2*X1^4*X2*X3^2*X4^2 + X1^4*X2*X3^2*X4 + 4*X1^4*X2*X3^2 + 3*X1^4*X2*X3*X4^3 + X1^4*X2*X3*X4^2 + 2*X1^4*X2*X3*X4 + 4*X1^4*X2*X4^2 +
4*X1^4*X2 + X1^4*X3^4*X4^4 + 3*X1^4*X3^3*X4^3 + X1^4*X3^3*X4^2 + X1^4*X3^2*X4^3 + 4*X1^4*X3^2*X4^2 + 4*X1^4*X3^2*X4 + 4*X1^4*X3^2 + 4*X1^4*X3*X4^2 +
4*X1^4*X3 + 4*X1^4*X4^2 + 4*X1^4*X4 + X1^4 + X1^3*X2^4*X3^4*X4 + 4*X1^3*X2^4*X3^3*X4^2 + 3*X1^3*X2^4*X3^3 + 4*X1^3*X2^4*X3^2*X4^3 + 2*X1^3*X2^4*X3^2*X4 +
X1^3*X2^4*X3^2 + X1^3*X2^4*X3*X4^4 + 2*X1^3*X2^4*X3*X4^2 + 3*X1^3*X2^4*X3*X4 + 3*X1^3*X2^4*X4^3 + X1^3*X2^4*X4^2 + 4*X1^3*X2^3*X3^4*X4^2 +
3*X1^3*X2^3*X3^4 + 3*X1^3*X2^3*X3^3*X4+ X1^3*X2^3*X3^3 + 4*X1^3*X2^3*X3^2*X4^4 + X1^3*X2^3*X3^2*X4^2 + 3*X1^3*X2^3*X3^2*X4 + X1^3*X2^3*X3^2 +
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4*X1^3*X2^2*X3^4*X4^3 + 2*X1^3*X2^2*X3^4*X4 + X1^3*X2^2*X3^4 + 4*X1^3*X2^2*X3^3*X4^4 + X1^3*X2^2*X3^3*X4^2 + 3*X1^3*X2^2*X3^3*X4 + X1^3*X2^2*X3^3 +
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+ 4*X1^3*X2^2*X3*X4 + X1^3*X2^2*X4^4 + X1^3*X2^2*X4^3 + 3*X1^3*X2^2*X4^2 + X1^3*X2^2 + X1^3*X2*X3^4*X4^4 + 2*X1^3*X2*X3^4*X4^2 + 3*X1^3*X2*X3^4*X4 +
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4*X2^2*X3^2*X4^4 + 3*X2^2*X3^2*X4^3 + 4*X2^2*X3^2*X4^2 + 2*X2^2*X3^2*X4 + 4*X2^2*X3^2 + 4*X2^2*X3*X4^4 +2*X2^2*X3*X4^2 + 3*X2^2*X3*X4 + X2^2*X3 +
4*X2^2*X4^4 + X2^2*X4^3 + 4*X2^2*X4^2 + X2^2*X4 + 4*X2*X3^4*X4^2 + 4*X2*X3^4+ 2*X2*X3^3*X4^3 + 2*X2*X3^3 + 4*X2*X3^2*X4^4 + 2*X2*X3^2*X4^2 + 3*X2*X3^2*X4 +
X2*X3^2 + 3*X2*X3*X4^2 + 3*X2*X3*X4 + 4*X2*X4^4 + 2*X2*X4^3 + X2*X4^2 + 4*X3^4*X4^2 + 4*X3^4*X4 + X3^4 + 2*X3^3*X4^3 + X3^3*X4^2 + 2*X3^3*X4 + 4*X3^3 +
4*X3^2*X4^4 + X3^2*X4^3 + 4*X3^2*X4^2 + X3^2*X4 + 4*X3*X4^4 + 2*X3*X4^3 + X3*X4^2 + X4^4 + 4*X4^3
```

4th Summation Polynomial S_4

$$
\begin{aligned}
S_4(x_1, x_2, x_3, x_4) &= \mathrm{Res}_Y(S_3(x_1, x_2, Y), S_3(x_3, x_4, Y)) \\
&= \det \begin{pmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 \\ 0 & b_2 & b_1 & b_0 \end{pmatrix} \\
&= a_2(b_0(a_2 b_0 - 2a_0 b_2 - a_1 b_1) + a_0 b_1^2) \\
&\quad + b_2(a_1(a_1 b_0 - a_0 b_1) + a_0^2 b_2).
\end{aligned}
$$

where $S_3(x_1, x_2, Y) = a_2 Y^2 + a_1 Y + a_0$
and $S_3(x_3, x_4, Y) = b_2 Y^2 + b_1 Y + b_0$.

We can evaluate this using only 21 multiplications and 24 additions.

Evaluating the Summation Polynomials

Theorem
Evaluating S_m at a point $(x_1, \ldots, x_m)$ in $\mathbb{F}_p^m$ can be done in $O(\log^2 p)$ steps for $m \ll p$.

Idea:
Use $S_m(x_1, \ldots, x_m) = Res_Y(S_3(x_1, x_2, Y), S_{m-1}(x_3, \ldots, x_m, Y))$ for $m \geq 4$ and $x_1, \ldots, x_m \in \mathbb{F}_p$.

This allows us to evaluate S_9 and S_{10} even if we can't compute them.

Our Variant of Index Calculus for the ECDLP

Let $P, Q \in E(\mathbb{F}_p)$ such that there exists k that solves $Q = kP$.

Factor Base (see also Amadori, Pintore, Sala 2017)

Compute random integers $a_1, ..., a_s$, $b_1, ..., b_s$. Then our factor base is $\mathcal{F} = \{a_1 P + b_1 Q, ..., a_s P + b_s Q\}$.

Find A Relation

Choose a multiset $\{x_1, \ldots, x_m\}$ with each $x_i \in \{x | (x, y) \in \mathcal{F}\}$ and check if $S_m(x_1, \ldots, x_m) = 0$.

Solving Step (see also APS17)

If $S_m(x_1, \ldots, x_m) = 0$ for some $\{x_1, \ldots, x_m\}$, then there exist y_i such that $(x_1, y_1) + \ldots + (x_m, y_m) = \mathcal{O}$ with (x_i, y_i) or $-(x_i, y_i)$ in $\mathcal{F}$. Substituting each $\pm(x_i, y_i)$ with $\pm(a_i P + b_i Q)$, we get a relation of the form $\sum_{i=1}^{m} \pm a_i P + \sum_{i=1}^{m} \pm b_i Q = \mathcal{O}$ and can solve for the discrete logarithm of Q, provided $\sum_{i=1}^{m} \pm b_i$ is invertible modulo the order of E.

Complexity

Theorem

The complexity of our algorithm is $O(p \log^2 p)$ *for* $m \ll s$ *and* $m \ll p$ *and* $\frac{s^{m-1}}{m!} \geq \log p$.

Still exponential, but better than other index calculus type algorithms for the ECDLP over prime fields which use Gröbner bases.

Our algorithm is embarrassingly parallel.

Another variant

Instead of evaluating S_m, choose a multiset of $m-1$ points from the factor base and check if the sum of those points is in the factor base.

Theorem

The complexity of this algorithm is $O(p)$ for $m \ll s$ and $s \geq (m-2)\log^2 p$.

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