On a conjecture of Erdös on 3-powerful numbers

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A conjecture of P. Erdös says that the diophantine equation $x + y = z$ has infinitely many solutions with $(x, y) = 1$ and such that if a prime p divides xyz, then p^3 divides xyz. In this paper, we give a proof of this conjecture.

Let $k \geq 2$ be an integer. A k-powerful number is a positive integer x with the property that if the prime p divides x, then p^k divides x (2-powerful numbers were called initially powerful numbers by S. W. Golomb [2]). In [1], P. Erdös studied k-powerful numbers and stated the following conjecture for $k = 3$ (see also [3], B 16):

Conjecture 1. (Erdös, 1975). *The diophantine equation*

$$x + y = z,$$

has infinitely many solutions in relatively prime 3*-powerful numbers.*

In this paper, we give a proof of this conjecture. In fact, we give an explicit construction of three infinite families of solutions.

Let a be a positive integer. Let x_0, y_0 and z_0 be relatively prime integers such that $x_0^3 + y_0^3 = az_0^3$. Define the sequence of triples (x_n, y_n, z_n) with the recurrences

$$\left\{ \begin{array}{l} x_{n+1} = x_n \left(x_n^3 + 2y_n^3 \right), \\ y_{n+1} = -y_n \left(2x_n^3 + y_n^3 \right), \\ z_{n+1} = z_n \left(x_n^3 - y_n^3 \right), \end{array} \right. \tag{1}$$

for $n \geq 0$. By induction one may show that $x_n^3 + y_n^3 = az_n^3$ for all $n \geq 0$. If we choose a, x_0, y_0 and z_0 such that

$$\left\{ \begin{array}{l} a \quad \text{is cube-free}, \\ (x_0, y_0) = 1, \\ x_0 y_0 \equiv 1 \pmod 2, \\ x_0 \equiv 2 \pmod 3, \\ y_0 \equiv 1 \pmod 3, \\ a | z_0^3, \end{array} \right. \tag{2}$$

then we have the following.

Theorem. *Let a, x_0, y_0 and z_0 be as above. For $n \geq 1$, let the triples (x_n, y_n, z_n) of integers be defined by recurrences (1). Then for all $n \geq 1$, $(x_n, y_n) = 1$, $a|z_n^3$ and $\lim_{n\to\infty} |z_n| = +\infty$.*

Proof. It is obvious that for all $n \geq 1$, x_n and y_n satisfy $x_n y_n \equiv 1 \pmod 2$, $x_n \equiv 2 \pmod 3$ and $y_n \equiv 1 \pmod 3$. Suppose that $(x_n, y_n) = 1$ and let p be a prime such that p divides $(x_{n+1}, y_{n+1}) = \left(x_n^3 + 2y_n^3, 2x_n^3 + y_n^3\right)$. Then p divides $2\left(x_n^3 + 2y_n^3\right) - \left(2x_n^3 + y_n^3\right) = 3y_n^3$ and hence $p = 3$, which contradicts the congruence $x_n^3 + 2y_n^3 \equiv 1 \pmod 3$. Then $(x_{n+1}, y_{n+1}) = 1$ and $z_n \neq 0$. Now, since $a|z_n^3$, then $a|z_{n+1}^3$. Finally, for all $n \geq 1$, $2^n|z_n$. Hence $\lim_{n\to\infty} |z_n| = +\infty$. ∎

If we start with integers a, x_0, y_0 and z_0 satisfying (2) and if we put for all $n \geq 1$,

$$\left\{\begin{array}{l} X_n = \min(|x_n|^3, |y_n|^3, a|z_n|^3), \\ Z_n = \max(|x_n|^3, |y_n|^3, a|z_n|^3), \\ Y_n = Z_n - X_n, \end{array}\right. \tag{3}$$

then X_n, Y_n and Z_n are relatively prime 3-powerful numbers. Thus each of the triplets (X_n, Y_n, Z_n) gives a solution of Erdös' conjecture.

Numerical examples.

(i) Let $a = 6$, $x_0 = 1,805,723$, $y_0 = -2,237,723$ and $z_0 = -960,540$. Then

$$x_0^3 + y_0^3 = -2^7 \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 2287^3.$$

(ii) Let $a = 9$, $x_0 = -271$, $y_0 = 919$ and $z_0 = 438$. Then

$$x_0^3 + y_0^3 = 2^3 \cdot 3^5 \cdot 73^3.$$

(iii) Let $a = 12$, $x_0 = 63,963,143$, $y_0 = -26,919,143$ and $z_0 = 27,226,290$. Then

$$x_0^3 + y_0^3 = 2^5 \cdot 3^4 \cdot 7^3 \cdot 13^3 \cdot 9973^3.$$

In these three examples, the integers a, x_0, y_0 and z_0 satisfy (2), and hence, using the recurrence relations (1), we get infinitely many solutions for the conjecture of Erdös. We see that in all these solutions two terms are perfect cubes. For the remaining cases, we state the following:

Conjecture 2. *The diophantine equation*

$$x + y = z,$$

has infinitely many solutions in relatively prime 3-powerful numbers none of which is a perfect cube or only one of them is a perfect cube.

References

1. P. Erdös, 'Problems and results on consecutives integers', *Eureka* 38 (1975−76), 3−8.
2. S. W. Golomb, 'Powerful numbers', *Amer. Math. Month.* 77 (1970), 848 − 852.
3. R. K. Guy, *Unsolved Problems in Number Theory*, Springer Verlag, New York, Heidelberg, Berlin, 1981.

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